**Relativistic Quantum Mechanics**

As we know, any theory must be compatible with Einstein’s theory of special relativity. This was recognized by Schrodinger and Heisenberg, etc., when they were trying to develop quantum mechanics, and they attempted from the first to make it fully compatible with special relativity. But they were unsuccessful with this, and only later realized that their formulations of quantum mechanics were none-the-less quite accurate in non-relativistic domains. It remained for Paul Dirac to develop a satisfactory relativistic formulation three years later in 1929. The usefulness of the Dirac equation is that it:

(1) predicts the spin states of the electron from first principles (so far we’ve had to add the spin degrees of freedom to the Hilbert space ad hoc),

(2) it predicts the (almost correct value of the) anomalous g-factor moreover appearing in the Hamiltonian for a spin in a magnetic field,

(3) and it correctly predicts the fine-structure corrections to the Hydrogen energy spectrum.

It also has some issues though, which we’ll get to in time.

**Criteria for relativistic ‘Schrodinger’ equation**

This is what we want out of any relativistic ‘Schrodinger’ equation. For one, we want it to be relativistically invariant, meaning when we make a Lorentz change of variables, the equation is invariant. And secondly we want the equation to naturally incorporate a particle’s spin degrees of freedom into the equation, and we’d also like an equation which predicts the anomalous g-factor appearing in the electron’s magnetic moment. We’ll explore these two a little more below.

**Relativistic invariance**

We need Lorentz invariance because according to Einstein, the laws of physics are invariant with respect to any inertial frame. The Schrodinger equation as presently written is not invariant. For example, let’s take a look at the free-particle Schrodinger equation in 1D (in position space)



Now let’s make a Lorentz change of variables to a primed reference frame moving at speed v to the right. The coordinates in the primed frame are related to the stationary frame coordinates by:





So filling these in we have the Schrodinger equation in the primed coordinates to be:



which is not the same equation by any stretch, though we will note that it is in the case where β → 0 and γ → 1 which is the classical limit. So why does it fail? Basically because the Schrodinger equation treats the time and position variables differently. The equation is first order in t and second order in x. So in order to make the equation relativistically invariant we will need an equation which treats both variables to the same order.

**Incorporation of spin d.o.f.**

Secondly we want the equation to naturally incorporate spin degrees of freedom instead of making us include it ad hoc as we’ve done so far. Mathematically, this means that we would like the relativistic ‘Schrodinger’ equation to be a 2 component matrix equation. And hopefully, when we incorporate the magnetic field into the equation, the g-factor will come out naturally.

**Klein-Gordon Equation**

Let’s go back to the Schrodinger equation – free particle for now,



So far we have been using the classical expression for the energy of a particle. Perhaps if we use instead the actual relativistic expression for the energy we’ll have our desired equation. So we would write,



This looks promising perhaps, but it still wouldn’t be relativistically invariant, as we can tell when we write it in position space,



because it still treats **r** and t at different orders. Plus, I’m not sure how mathematically well defined the √operator is. One possibility is to square both sides of the equation. When we do we get the so-called Klein-Gordon equation:



If we put this into position space, we’ll have:



which we can write as:



If we use the metric ημν = (+1, -1, -1, -1), as is convention in relativistic quantum mechanics/quantum field theory, then we may write this as:



Where we recognize ∂2 as the wave-operator, box, or whatever you want to call it, and as the magnitude of the SR gradient (see Special Relativity tensor stuff in Classical Mechanics):



Since,



As a scalar, it is invariant, and so the KG equation is relativistically invariant as required. Let’s be a little more explicit. So when we change to a new reference frame ψ(**x**,t) --> ψ´(**x**´,t´), where ψ´ is reference to fact that wavefunction may not be a scalar and so could have a different value at the (equivalent) points (**x**,t) and (**x**´,t´). But we presuppose that ψ is a scalar and so ψ´(**x**´,t´) = ψ(**x**,t). Also, note that ψ(**x**,t) is just ψ´(**x**´,t´) but expressed in the new coordinate system where it’s now a function of **x** and t, rather than **x**´ and t´. So repeating, starting from the primed system and going back to the unprimed system…



From either bottom or second from bottom line, can see that we have η being transformed to the old coordinate system. And this is just η back again, since it’s a tensor. So we have:



So that’s good, but there is a major problem. Consider our interpretation of the wavefunction. We normally would say that ρ = ψ\*ψ, and **j** = Re(ψ\*ψ). However we identify ρ and **j**, they must be related via the continuity equation:



But if we define ρ as ψ\*ψ, one can see that we won’t be able to satisfy this form. Instead, we can define:



Then consider the time derivative,



We can fill in the KG equation,



and so identify:



The ρ is unlike what we’re accustomed to, though **j** is familiar. But the problem here is that there is no guarantee that ρ will actually be positive, thanks to the minus sign. And so it cannot be interpreted as a probability density. This was deemed a fatal flaw, though it turns out the equation still finds use in QFT. Another problem though is that it doesn’t force a two – component matrix representation of a particle upon us. As a consequence, the equation cannot describe a spin-full particle. It would be restricted to describing spin-0 bosons, which it does quite well in QFT context. But its not what we want since we want to be able to describe fermions.